UDC 534.222.2 : 539.374

ON THE STRUCTURE OF SHOCK WAVES IN ELASTIC-PLASTIC MEDIA

PMM Vol. 37, №2, 1973, pp. 300-305

Ia. A. PACHEPSKII
(Moscow)

(Received June 7, 1971)

A system of equations of the shock structure in an elastic-plastic medium is considered. It is assumed that volume strain occurs elastically, and the shear strain equations are a combination of Hooke's law in differential form, and the plastic flow law with a plasticity limit which is a nondecreasing function of the pressure [1]. The assumption that the properties of the thermodynamic functions for reversible processes can be carried over to the irreversible case permits making a number of deductions about the solutions of the structure equations and the character of the singularities of these equations. Conditions are obtained which the thermodynamic functions of the material, and the initial and final states should satisfy in order for shock structure to exist. Such an analysis of the shock front structure is contained in [2] for media with a global stress tensor. The considerations of [2] are extended naturally to the case of the existence of a unique stress-strain dependence for the uniaxial strain of solids.

The system of equations describing the behavior of a material in which the volume strain is purely elastic but part of the infinitesimal shear strain can occur plastically contains the equations expressing the conservation laws and the shear strain equation [1]. Hence, the system of equations of plane one-dimensional stationary motion of an elastic-plastic material with viscosity and heat conductivity taken into account is

$$\frac{d}{dx}(\rho u) = 0, \quad \frac{d}{dx}\left(-\sigma + \rho u^2 - \mu \frac{du}{dx}\right) = 0, \quad \sigma = -p + q$$

$$\frac{d}{dx}\left[\rho u \left(E + \frac{u^2}{2}\right) - \sigma u - \mu u \frac{du}{dx} - \varkappa \frac{dT}{dx}\right] = 0$$

$$u \frac{dq}{dx} + \lambda q = {}^4/_3 G \frac{du}{dx}, \qquad e(y) = \begin{cases} 1 & y \geqslant 0 \\ 0 & y < 0 \end{cases} \tag{1}$$

$$\lambda = \frac{1}{2h^2(p)}\left({}^4/_3 Gq \frac{du}{dx} - uhh' \frac{dp}{dx}\right) e\,[q^2 - h^2(p)]\, e\left\{q\left[{}^4/_3 G \frac{du}{dx} + (-1)^{r+1} h' \frac{dp}{dx}\right]\right\}$$

Here p is the hydrostatic pressure, σ is the normal stress, $q = \delta + p$ the value of the deviator component of the stress tensor on an area with normal directed along the flow axis towards increasing x; G is the shear modulus of the material, $\varkappa$ the thermal conductivity, E the specific inner energy, u the particle velocity in the coordinate system connected to the shock front. A dissipative mechanism analogous to a viscous mechanism is given by the Landau dissipation tensor [3]

$$\sigma_{ij}' = 2\eta\,(e_{ij} - {}^1/_3\delta_{ij}e_{kk}) + \zeta e_{kk}\delta_{ij}, \quad \mu = {}^4/_3\eta + \zeta$$

where η and ζ are the first and second coefficients of viscosity. The plasticity condition is

$$J = {}^1/_2 s_{ij} s_{ij} = {}^3/_4 h^2(p)$$

Here J is the second invariant of the stress tensor, $h'(p) \geqslant 0$; we assume that $h(p) = h_r(p)$, $r = 1$ $(r = 2)$ if plasticity is by shear with compression (with rarefaction), $q = (-1)^r h_r(p)$ in the state of plasticity by shear. We consider the viscous and plastic dissipation to occur in parallel.

Let us integrate once in the system (1) by assuming that the gradients of all the quantities tend to zero as $x \to -\infty$ and $x \to \infty$, and the values of the quantities themselves equal the values ahead of and behind the shock-wave front, respectively. Denoting the stream density by j, we have

$$u = jv, \quad \mu j \frac{dv}{dx} = p - q + j^2 v - \alpha = \Pi(v, p, q)$$

$$\varkappa \frac{dT}{dx} = j(E - {}^1/_2 j^2 v + \alpha v - \beta) = jF(v, p, q) \frac{dq}{dx} + \frac{\lambda_1}{v} q = \frac{4G}{3v}\frac{dv}{dx}, \qquad (2)$$

$$\lambda_1 = \frac{1}{2h^2(p)}\left({}^4/_3 Gq\frac{dv}{dx} - vhh'\frac{dp}{dx}\right) e\,[q^2 - h^2(p)]\, e\left\{q\left[{}^4/_3 G\frac{dv}{dx} + (-1)^{r+1} h' v\frac{dp}{dx}\right]\right\}$$

$$\alpha = p_0 - q_0 + j^2 v_0 = p_1 - q_1 + j^2 v_1 \qquad (3)$$

$$\beta = E_0 - v_0(p_0 - q_0) + {}^1/_2 j^2 v_0 = E_1 - v_1(p_1 - q_1) + {}^1/_2 j^2 v_1$$

Here α, β are the constants of integration, and the subscript zero refers to the state ahead the shock and one to the state behind the shock. The shocks are obtained as the limit of the flows under consideration.

In contrast to gasdynamics, in one-dimensional flows, the state of the substance is characterized by values of not two, but three quantities, the shear and normal stresses and the density, say, or p, q, v. Hence, even in the case of a single shock crossing two relations on the shock (3) are not sufficient to determine the state behind the shock by means of data on its intensity and state ahead of the shock. As will be clear from the sequel, considerations of stability of the shock crossing are also insufficient to isolate a single shock crossing.

The fundamental thermodynamic identity for reversible processes is

$$TdS = dE + pdv - \frac{v}{2G}\,dJ$$

Furthermore, let us consider E, T and the entropy S per unit mass to be functions of v, p, J (or v, p, q), $G = G(v, J)$. Since dE and dS are total differentials, we obtain

$$E_p = TS_p, \quad E_v = TS_v - p, \quad E_J = TS_J + g \quad (g = {}^1/_2 v / G)$$

$$T_p S_v - T_v S_p = 1, \quad T_J S_p = T_p S_J, \quad T_J S_v - T_v S_J = g_v \qquad (4)$$

Let us take a hypothesis analogous to that used in examining a viscous fluid that the relationships (4) are conserved for the irreversible case (for plastic strain). We assume below that $S_p > 0$ and $\partial G / \partial v \leqslant 0$, from which $S_J > 0$ follows.

Let us examine trajectories of the system (2) in the space of the variables v, p, q. We introduce the surfaces F, h_r $(r = 1, 2)$, Π, H given by the equations

$$F(v, p, q) = 0, \quad q = (-1)^r h_r(p), \quad \Pi(v, p, q) = 0$$

$$H = E - E_0 + {}^1/_2(\sigma + \sigma_0)(v_0 - v) = 0$$

Here H is the surface of the shock adiabat of the substance. Let Q_M denote the integral surface of the equation $dg / dv = {}^2/_3 g$ passing through the point M. The asterisk on a symbol will denote the partial derivative and means that the corresponding value is calculated either along some Q_M or along h_r $(r = 1, 2)$. Hence, for any functions $\Phi(v, p, q)$ in a domain of elasticity by shear $\Phi_v^* = \Phi_v + \Phi_J q / g$, $\Phi_p^* = \Phi_p$, and in the plastic flow domain $\Phi_v^* = \Phi_v$, $\Phi_p^* = \Phi_p + {}^3/_2 hh' \Phi_J$. In this notation an equation

$$\frac{dv}{\Pi} = \frac{T_p^* dp}{\mu_l^2 F / \varkappa - T_v^* \Pi} \tag{5}$$

can be obtained from the system (2).

The following can be asserted relative to the properties of the integral curves (2) and the properties of the surfaces F, Π, H (the subscript L denotes the derivative along the integral curve).

1. The sign of the quantities $(dp / dv)_L^* + T_v^* / T_p^*$ agrees with the sign of the product $F\Pi$; $(dv / dp)_L^*$ is only at points of Π not singular for (2).

2. The sign of the quantity $(d^2v / dp^2)_L^*$ agrees at points of Π with the sign of $F(v, p, q)$.

3. $(dp / dv)_L^* < 0$ where $F(v, p, q)\,\Pi(v, p, q) < 0$.

4. The quantity $(dp / dv)_L^* + F_v^* / F_p^*$ is positive at points of the surface F in the domain of elasticity by shear and on h_1 everywhere that $\Pi(v, p, q) < 0$, and can change the sign in the domain of elasticity by shear and on h_2 at points where $\Pi(v, p, q) > 0$. This quantity is positive in some neighborhood of the singularities in the domain of elasticity by shear.

5. The surfaces F and H intersect at points of the surface Π and only there. Outside of Π the surfaces F and H lie on one side of Π.

6. $T(dS / dv)_F^* = \Pi(v, p, q)$ and $dS = TdS$ at points of Π.

A section of the integral curve (5) corresponds to compression (rarefaction) if it lies under (over) Π.

Let us investigate the singularities of Eq. (5) which are solutions of the system:

$$F(v, p, q) = 0, \; \Pi(v, p, q) = 0, \; -h_1(p) \leqslant q \leqslant h_2(p)$$

by assuming that $h_r' < 1$. We consider that the singularity lies together with its neighborhood either in the domain of shear elasticity or in the domain of shear plasticity. In conformity with [4], the investigation of the singularities of (5) can be reduced to an investigation of the singularities of the first approximation system if the determinant of the matrix of the first approximation system is not zero (in this case $Y = \Pi_v^* S_p^* - \Pi_p^* S_v^*$). The quantity has the same sign as the scalar product of the normal to Π at the tangent to a section of the surface Γ by the surface Q_M or h_r directed towards increasing v.

If $Y \neq 0$ at the singularity of (5), then the following can be proved:

1. The singularity of (5) is a node if $Y > 0$ and a saddle if $Y < 0$.
2. A node-type singularity for (5) is always unstable.
3. For exclusive directions $z_{1,2}$ the inequalities

$$-S_v^* / S_p^* < z_1 < -\Pi_v^* / \Pi_p^* < z_2 < \infty$$

are valid at the singular point in the case of a saddle, and

$$-\Pi_v^* / \Pi_p^* < z_1 < -S_v^* / S_p^* < z_2 < \infty$$

in the case of a node.

4. Let the domain D consist of points at which the inequality $F(v, p, q)\,\Pi(v, p, q) < 0$ is satisfied. Then one of the exclusive directions always enters the domain $\bar{D}$, where this exclusive direction is stable for the saddle case.

5. For exclusive directions entering $\bar{D}$ the approach to the saddle singularity is always possible on h_2, and is known to be possible on h_1 if the inequality $S_p + {}^3/_2 h_1 h_1' S_J > {}^3/_2 h' g S_v$ is satisfied at the singularity.

The following assertions are valid for the case $Y = 0$.

6. The exclusive directions are

$$z_1 = -\Pi_v^* / \Pi_p^*, \qquad z_2 = (\mu j^2 S_p / \varkappa - T_v^* \Pi_p^*) / T_p^* \Pi_p^*$$

7. An investigation of the nature of the singularity for $Y = 0$ reduces to an investigation of the nature of the tangent to the surfaces Π and F at this point. Let

$$R^{(i)} = \left(\frac{\partial^i p}{\partial v^i}\right)_F^* - \left(\frac{\partial^i p}{\partial v^i}\right)_\Pi^* = 0 \quad (i = 1, 2, \ldots, n-1), \qquad R^{(n)} \left(\frac{\partial^n p}{\partial v^n}\right)_F^* - \left(\frac{\partial^n p}{\partial v^n}\right)_\Pi \neq 0$$

Then if n is odd and $R^{(n)} > 0$ ($R^{(n)} < 0$), the singularity is a node (saddle). If n is even and $R^{(n)} > 0$ ($R^{(n)} < 0$), then upon approaching from the side $v < v_0$ the trajectories will behave as at a saddle (nodal) point, and upon approaching from the side $v > v_0$ as at a noddle (saddle) point.

Let us examine some singularity M of the system (2). If the inequality $h_2(p_M) > q_M > -h_1(p_M)$ is satisfied for the point M, then its nature for (2) is the same as for (5). If M lies on h_r, then it should be characterized twice: as a singularity of (5) in the domain of shear elasticity, and as a singularity of (5) in the domain of plastic shear.

For any two singularities M_k and M_l of the system (2) a curve can be constructed connecting M_k and M_l which we call Π and denote by Π_{kl}. To construct it we draw Q_M through each of the singular points M_k, M_l. If the shock crossing $M_k \to M_l$ is considered, then Π_{kl} consists of segments of the intersections Q_{M_k}, Q_{M_l} with Π (one or both can be missing if one or both singularities lie on h_r) and segments of the intersections between Π and h_r, where r is selected so that for motion from M_k to M_l over h_r along Π_{kl} the domain of shear elasticity is on the right.

Assuming that the approach along the exclusive direction entering D to the singularity corresponding to the final state is possible, the following assertions can be proved:

1) Let there be just one breakpoint on Π_{01} corresponding to the transition from elasticity to plasticity by shear for the motion $M_0 \to M_1$ or let there not be such a break. If there are points on Π_{01} where $F(v, p, q) = 0$, then no continuous solution of (2) corresponding to the shock $M_0 \to M_1$ exists. In considering rarefaction shocks we hence assume that the temperature on the section between M_0 and M_1 of the curve of the intersection of F with the same surfaces with which Π intersects to form Π_{01}, is everywhere less than in at least one of the points of Π_{01}.

2) Let there be one breakpoint on Π_{01} corresponding to the transition from elasticity to plasticity during motion from M_0 to M_1, or let there not be such a break. Let $F(v, p, q)(v_0 - v) \geqslant 0$ on Π_{01} everywhere, and let the condition formulated in (1) for the temperature be satisfied for the case of a rarefaction shock. Then there exists a sequence of continuous solutions of (2) corresponding to a sequence of shock crossings of the considered intensity which transfers the substance from the state M_0 into the state

M_1. If F hence has no common points with Π, then there exists a unique continuous solution of (2) corresponding to the shock crossing $M_0 \to M_1$.

3) A singularity of (2) lying on h_2 and being a saddle point for (5) in the shear plasticity case is also a saddle point in the shear elasticity case.

4) A sequence of shock crossings carrying over $M_0 \to M_1$ is impossible if $v_0' < v_1'$ (v_i' is the specific volume at the intersection of Q_{M_i} with the v axis) and $M_1 \notin h_2$.

5) Let $Q_{M1} \not\equiv Q_{M_0}$, $M_1 \notin h_1$, $v_1' < v_0'$. Then for the existence of a sequence of shock crossings $M_0 \to M_1$ it is necessary and sufficient that a singularity M_2 exists on h_1 such that the condition (3) would be satisfied for the pair of points M_0M_2 and M_2M_1.

6) If singularities on h_1 which are saddle points for (5) in plastic shear are also saddle points for elastic shear, then compression shock crossings $M_0 \to M_1$ are possible only in states corresponding to the points M_1 on h_1 or in states corresponding to points M_1 such that $Q_{M_0} \equiv Q_{M_1}$.

Let us note that sufficient for the compliance with condition (6) is the validity of the inequality

$$^3/_2\, S_j h_1 (^3/_2\, g h_1' j^2 - 1) > {}^3/_2\, g h_1' S_v - S_p \tag{6}$$

or the stronger constraint $h\,(p) < {}^4/_3 G \,/\, (1 - \partial \ln G \,/\, \partial \ln v)$ at the singularities.

The conditions $Y \geqslant 0$ and $Y \leqslant 0$ which must be satisfied for the initial and final states is none other than the condition for shock front stability. Let us note that these conditions can be satisfied even in the presence of the constraint (6) at points of the curve of intersection of F and Π in the domain of shear elasticity.

When a continuous solution of (2) exists which corresponds to the shock crossing $M_0 \to M_1$ with compression (rarefaction), the temperature along the integral curve increases (decreases) monotonously, and the entropy has an absolute maximum within the flow.

An examination of the shock wave structure permits supplementing the system (3) by the missing boundary conditions in order to define the uniqueness of the solution (3) (in the case of the existence of a unique shock crossing). In fact, if the shock crossing is accomplished in the domain of shear elasticity, then M_1 lies on $Q_{M'}$, where M' is the point from which the shock crossing is completed at M_1; if the shock crossing is accomplished in the domain of shear plasticity, then $q_1 = (-1)^r\, h\,(p)$. Let specific dependences $G\,(v, J)$, $E\,(v, p, q)$, $S\,(v, p, q)$ and the initial state of the medium characterized by the point M_0 be given. In order to investigate the possible shock crossings from the point M_0 it is convenient to construct a curve $\sigma_H\,(v, v_0)$ in the $\sigma - v$ plane which is the dependence of σ on v on the line of intersection of H with Q_{M_0} and with $h_r\,(r = 1, 2)$ at such points M at which $(-1)^r\, v'\,(M) \geqslant (-1)^r v'\,(M_0)$. In the shear elasticity domain $\sigma_H\,(v, v_0)$ is given by the equality

$$E\,[v,\, -\sigma + q_*\,(v, v_0),\; q_*\,(v, v_0)] - E_0 = \frac{\sigma + \sigma_0}{2}\,(v - v_0)$$

Here $q_*\,(v, v_0)$ is the solution of the equation $dq \,/\, dv = {}^4/_3 G \,/\, v$ for $q\,(v_0) = q_0$. In the shear plasticity domain $\sigma_H\,(v, v_0)$ can be obtained from

$$E\,\{v,\; p_*\,(\sigma),\; (-1)^r h_r\,[p_*\,(\sigma)]\} - E_0 = \frac{\sigma + \sigma_0}{2}\,(v - v_0)$$

where $p_*\,(\sigma)$ is given implicitly by the equation $\sigma + p = (-1)^r\, h_r\,(p)$. The quantity Π is mapped on the $\sigma - v$ plane by the ray $\Pi_\sigma = \sigma - \sigma_0 + j^2\,(v - v_0) = 0$.

Let us examine the point of intersection of Π_σ and $\sigma_H(v, v_0)$. Corresponding to the possible shock crossings will be those points M_1 of the intersection of Π_σ and $\sigma_H(v, v_0)$ for motion to which along the ray from M_0 there are no points of σ_H to the left of the ray. Furthermore, if M lies in the shear plasticity domain with compression, then the condition $Y(M) < 0$ should be satisfied, where the calculations should be carried out for shear elasticity. If this condition is satisfied, then the shock crossing is unique. In particular, it is satisfied if (6) is satisfied. If $Y(M) \geqslant 0$ for shear elasticity, then a shock crossing of the same intensity is possible with the change in the character of the shear strain. This shock crossing can only be by a rarefaction shock crossing. To seek the state which is final, a branch of the curve $\sigma_H[v, v(M)]$ corresponding to rarefaction can be constructed on the $\sigma - v$ plane. Its point of intersection with the ray Π_σ corresponds to the final state of the second shock crossing.

The author is grateful to S. S. Grigorian for supervising the research and to G. Ia. Galin for useful discussion.

BIBLIOGRAPHY

1. Grigorian, S. S., On basic concepts in soil dynamics. PMM Vol. 24, №6, 1960.
2. Galin, G. Ia., On the theory of shock waves. Dokl. Akad. Nauk SSSR, Vol. 127, №1, 1959.
3. Landau, L. D. and Lifshits, E. M., Theoretical Physics, Vol. 7. Theory of Elasticity, Published in English, Pergamon Press, 1959.
4. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations, Gostekhizdat, Moscow-Leningrad, 1949.

Translated by M. D. F.

UDC 539.31 : 539.375

STRESS CONCENTRATION ON AN ELLIPSOIDAL INHOMOGENEITY IN AN ANISOTROPIC ELASTIC MEDIUM

PMM Vol. 37, №2, 1973, pp. 306-315
I. A. KUNIN and E. G. SOSNINA
(Novosibirsk)
(Received April 26, 1972)

The tensor stress concentration coefficient connecting the stress on the boundary of an inhomogeneity in an anisotropic elastic medium with the external field is represented as the product of two factors. The first is universal for any inhomogeneity and depends on the elastic constants of the medium and the inhomogeneity, and on the normal to the surface. Its construction reduces to an algebraic operation of inverting a third-order matrix. The second factor is a constant tensor in the ellipsoid case, which is expressed in terms of the mean value of the first factor over the surface of the ellipsoid. Explicit formulas are obtained from the homogeneous and linear external fields. The cases of a cavity and rigid inclusion are examined separately. For an arbitrary polynomial field the problem